Analytical solution for the Feynman ratchet

Karol Pesz,[1,*] Barbara J. Gabryś,[2,†] and Stanisław J. Bartkiewicz[1]
[1]*Institute of Physical and Theoretical Chemistry, Technical University, Wybrzeże Wyspiańskiego 27, Wrocław, Poland*
[2]*Faculty of Mathematics and Computing, The Open University in the South, Boars Hill, Oxford OX1 5HR, United Kingdom*
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A search for an analytical, closed form solution of the Fokker-Planck equation with periodic, asymmetric potentials (ratchets) is presented. It is found that logarithmic-type potential functions (related to ‘‘entropic’’ ratchets) allow for an approximate solution within a certain range of parameters. An expression for the net current is calculated and it is shown that the efficiency of the rocked entropic ratchet is always low.

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I. INTRODUCTION

In the last decade there has been a lot of interest in phenomena where concerted action of randomness and causality come into play. A large body of work exists in the area of fluctuation-induced directed transport: the gist of the matter is whether thermal fluctuations can help extract energy from stochastic processes, which is by no means obvious. In most cases, no matter how sophisticated the mathematics and physics involved, the principle behind the rectifying motion is that featured by Feynman's famous ratchet and pawl model [1]. The usefulness of the prototype models of the ratchet effect becomes apparent when considering wide-ranging applications in modern biology and nanotechnology. These are modeling molecular motors, explanation of ‘‘power strokes’’ in muscles, rectification of motion from random movements, improvements of performance of superconducting materials, and motion of colloids in asymmetrically structured channels [2–7].

In accordance with the second law of thermodynamics, usable work cannot be extracted in equilibrium conditions in a spontaneous process regardless of how sophisticated a device we design. However, when the common features of ratchetlike devices are considered, it is well established that with a kind of broken symmetry (space, time, or both) operating in the presence of nonequilibrium conditions (chemical reaction, external perturbation, energy dispersion, state-dependent diffusion, correlated noises, etc.), directed transport can emerge as an intrinsic phenomenon.

In general, the stochastically boosted unidirectional transport is related to one of the three main kinds of fundamental mechanisms: (i) a competition between state-dependent diffusivity $D(x)$ and force $F(x)$, (ii) external bias, and (iii) incommensurability. Büttiker [8], Magnasco [9], and Prost, Ajdari, and co-workers [10], respectively formulated simple theoretical models that lie behind each of these effects.

The earlier model by Büttiker [8] (see also Landauer [11]) makes use of periodic functions to describe a specific interplay of deterministic forces and diffusivity. In fact, this model concerns a quite delicate problem of the state- or space-dependent diffusion coefficient; sometimes the effect is referred to as a multiplicative noise. A spontaneous transport can appear as a result of a broken balance between periods of deterministic and diffusive terms. A proper choice of a potential $V(x)$ and diffusion $D(x)$ may result in an effective potential that contains a linear term, a source of a constant driving force. The primary mechanism for transport in the Büttiker model comes from a phase difference of otherwise perfectly periodic functions. When a potential function tries to localize probability distribution in a place, but diffusion opts for other places, the system, being frustrated, looks for a compromise and sends a current that forbids the solution to be unstable.

A real breakthrough in modeling ratchets was the seminal work by Magnasco [9]. The author solved a Fokker-Planck equation for a simple model for a piecewise potential barrier, and derived an exact formula for a net current flowing in a ‘‘rocked’’ one-dimensional system. In quantum mechanics one considers a sort of counterpart for the Magnasco model: it is the Schrödinger equation for a particle in a square potential well. However, the Schrödinger equation allows for use of a number of physically relevant, smooth potential functions, while the Smoluchowski equation used by Magnasco does not seem to lend itself to this approach easily [4,5]. Nonetheless, the piecewise barrier has become a standard in many approaches for the ratchet effect. It is easily implemented in numerical analyses; it allows one to calculate current as a function of an external perturbation; and it encompasses all important issues about the balance of diffusive and ballistic motions. Unfortunately, its drawback is to be rather unphysical.

While models based on that by Magnasco become ‘‘babies of the family,’’ as far as we know there have been no attempts to use smooth potential functions in order to solve the ratchet problem with periodic potentials. The present work attempts to contribute to this field, since having a kind of analytical solution is like having a reference marker. With strict solutions to hand one can reasonably compare the results of different numerical simulations, and estimate the quality of a simulation.

In this paper we present a potential function of a logarithmic type that offers a quite good analytical approximation without referring to stepwise forces. Potentials of a similar form appear in a natural way for ‘‘entropy’’ ratchets (when higher dimensions are taken into account) as is shown elegantly in a recent paper by Braun [12] (see also Ref. [13]).

*Deceased
†Corresponding author. FAX: +44-1865-736288. Electronic address: b.j.gabrys@open.ac.uk

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We believe it is one of the few attempts to date to find an approximation of an analytical solution (with an arbitrary accuracy, when necessary) for an asymmetric ratchet potential.

II. OUTLINE OF THE PROBLEM

We consider a simple model of a massless single particle diffusing along one dimension in the presence of asymmetric potential with period π, $V(x)=V(x+\pi)$. In order to refer to other studies of overdamped motion, we argue that such an approach is traditionally based on the reduced Langevin equation

$$\dot{x}=F(x)+\sqrt{D(x)}\xi(t), \tag{1}$$

where x stands for a position of a moving particle, the friction constant is set to 1, and the deterministic force $F(x) = -dV/dx$. Random forces $\xi(t)$ have properties of white noise, i.e., $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=2D_0\delta(t-t')$. In what follows we assume also no state-dependent diffusion, $D(x)=D_0$.

In general, this problem as stated in terms of the Langevin equation, can be converted into an analysis of the Fokker-Planck (or rather Smoluchowski) equation. Then, the task is to find the probability density function, $P(x,t)$,

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left\{D_0\frac{\partial P}{\partial x}-F(x)P\right\}. \tag{2}$$

Equation (2) can be written as a continuity equation, $\partial P/\partial t = -\partial J/\partial x$, with the density current

$$J=-D_0\frac{\partial P}{\partial x}+F(x)P. \tag{3}$$

For a steady state, with the probability being the function of position only, $P(x,t)=P(x)$, a nonzero current can flow as a result of balance between the regular force and diffusion-dependent terms.

In general, it is well known that potential forces try to localize particles in places where the potential function has minima. The probability of finding such minima scales with the Boltzmann factor $\exp[-V(x)/D_0]$, and all the main probability function features come from the shape of the potential.

An alternative mechanism for the ratchet transport is due to Magnasco [9]. This "rocked" ratchet is driven by a time-dependent external force $f(t)$ that may be represented as a slowly changing (square or sinusoidal) signal rocking the ratchet left and right. The external bias may be introduced into the Smoluchowski equation by simply adding its amplitude to the original force $F(x)$. Then, the equation for the current reads

$$J=-D_0\frac{\partial P}{\partial x}+[F(x)+f]P. \tag{4}$$

Note that in Eq. (4) the external force f is now time independent. Magnasco's approach, while convincing and elegant,

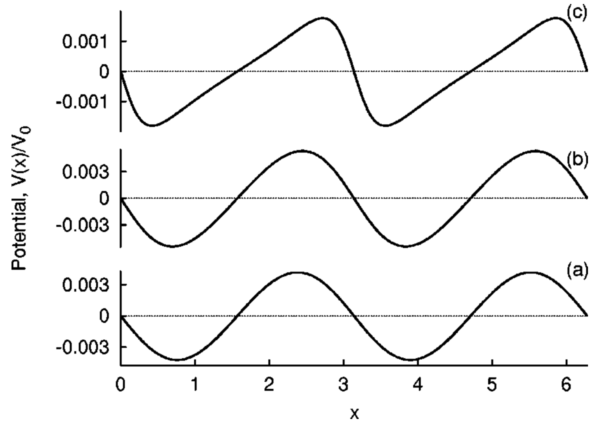

FIG. 1. A plot of the reduced potential function $V(x)/V_0$ ($V_0=10$) as a function of position x for different values of parameters. From bottom to top: (a) $\alpha=-0.1$, $\gamma=-0.1$; (b) $\alpha=-0.05$, $\gamma=-0.3$; (c) $\alpha=-0.1$, $\gamma=-0.8$.

treats this equation in two regions in which deterministic forces are simply constants of opposite signs and have significantly different values. In this paper we propose a different potential function (and mention a class of similar functions) that does not suffer from such violent jumps in force values. It is smooth, physically reliable, and (for a properly tuned parameters) analytically feasible.

In what follows we assume a potential function in the form of logarithm

$$V(x)=V_0\ln\{A+\tilde{G}(x)\}, \tag{5}$$

where V_0 is the amplitude of the potential, A is an arbitrary constant, and $\tilde{G}(x)$ introduces asymmetry and is similar to the equation for an asymmetric potential (A1) (see Appendix A).

In particular, we choose $\tilde{G}(x)$ in the form

$$\tilde{G}(x)=\tilde{\alpha}\frac{d}{dx}\ln[\tilde{\beta}+\tilde{\gamma}\cos^2(x)], \tag{6}$$

where $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are constants.

Such an asymmetric function can serve as a "core-potential" model since every monotonic function of $\tilde{G}(x)$ is also an asymmetric one. As any potential is always defined with an accuracy up to a constant, we can remove dependence of $V(x)$ on A and $\tilde{\beta}$ (or simply put their values equal to 1). In effect, the potential we are considering has the form

$$V(x)=V_0\ln\{1+G(x)\} \tag{7}$$

with

$$G(x)=\alpha\frac{d}{dx}\ln\{1+\gamma\cos^2(x)\}=-\frac{2\alpha\gamma\sin(x)\cos(x)}{1+\gamma\cos^2(x)} \tag{8}$$

depending on three parameters only. Examples of potential functions of this kind are presented in Fig. 1 for different values of the shape parameter. Note that an effective potential in the form of the logarithm of an oscillating function was recently derived by Braun [12] for entropic barriers that hinder the motion of atoms on a metal surface. Earlier Cecchi and Magnasco [13] referred to *entropic ratchets* in a construction of a two-dimensional asymmetric potential containing also a logarithmic term.

III. AN APPROXIMATE SOLUTION TO A SMOOTH RATCHET POTENTIAL

General methods of solving the Fokker-Planck equations are described in depth by Risken [14]. In our case, solving Eq. (4) with potential (7) yields the following probability density $P(x)$:

$$P(x)=H^{(-)}(x)\exp(\varphi x)\left\{P_0-j\int_0^x H^{(+)}(x')\times\exp(-\varphi x')dx'\right\}. \tag{9}$$

In Eq. (9) $H^{(\pm)}(x)=[1+G(x)]^{\pm\mu}$, $\mu=V_0/D_0$, $\varphi=f/D_0$, $j=J/D_0$, and P_0 stands for an initial value of this probability density.

There are no analytical solutions of Eq. (4) expressed in terms of elementary functions when $F(x)$ is obtained as a derivative of smooth periodic asymmetric potentials.

Here we propose a way of solving algebraic functions appearing in Eq. (9): we take advantage of a kind of the series expansion for limited values of parameters appearing in Eq. (9). Namely, we expand $H^{(\pm)}(x)=[1+G(x)]^{\pm\mu}$ in terms of the shape parameter γ when $|\gamma|<1$ and $|\alpha|<1$. Such an approach offers a rather good approximation for the exponentiated potential function, therefore one can expect that integrals calculated within this approximation are reasonably close to the exact solution. In principle, an exact solution can be represented in the form of an infinite convergent series.

The expansion of the integrating factor $H^{(-)}(x)$ reads

$$H^{(-)}(x)=1+\sum_{n=1}^{\infty} S_n\sin^n(2x)\times\left\{1+\sum_{m=n}^{\infty}\sum_{k=0}^{m-n+1} C_{nmk}\cos^k(2x)\right\}, \tag{10}$$

where

$$S_n=\frac{(\mu)_n}{n!}(\alpha\gamma)^n, \tag{10a}$$

$$C_{nmk}=\frac{m!(-\gamma)^{m-n+1}}{(n-1)!k!(m-n-k+1)!}, \tag{10b}$$

and $(\mu)_n$ stands for Pochhammer's symbol [15],

$$(\mu)_n=\mu(\mu+1)\cdots(\mu+n-1)=\frac{\Gamma(\mu+n)}{\Gamma(\mu)}. \tag{10c}$$

On the other hand, in the expansion (10) products and powers of trigonometric functions can be expressed in terms of higher harmonics of even multiples of x. Therefore, the series may have the following form:

$$H^{(-)}(x)=h_0^{(-)}+\sum_{k=1}^{\infty}\{s_{2k}^{(-)}\sin(2kx)+c_{2k}^{(-)}\cos(2kx)\}, \tag{11}$$

where

$$h_0^{(-)}=1+\frac{(\mu)_2}{4}\alpha^2\gamma^2\left\{1-\gamma+\frac{15}{16}\gamma^2-\frac{7}{8}\gamma^3+\frac{105}{128}\gamma^4\right\}+\alpha^4\gamma^4\frac{(\mu)_4}{64}\left\{1-2\gamma+\frac{35}{12}\gamma^2\right\}+\alpha^6\gamma^6\frac{(\mu)_6}{2304}. \tag{11a}$$

In passing from the form (10) to Eq. (12) a rather trivial algebra is involved, therefore we show explicitly only several coefficients $s_{2k}^{(-)}$ and $c_{2k}^{(-)}$ of the series (12) in Appendix B.

A similar kind of expansion can be performed for the term $H^{(+)}(x)$ [under the integral sign in Eq. (9)]. The functions $h_0^{(+)}$ and the coefficients $s_{2k}^{(+)}$ and $c_{2k}^{(+)}$ are easily obtained from $h_0^{(-)}$ and $s_{2k}^{(-)}$and $c_{2k}^{(-)}$ through the obvious change of $\mu\to-\mu$, respectively.

With the help of this expansion the integral appearing in Eq. (9),

$$\Psi(x)=\int_0^x H^{(+)}(x')\exp(-\varphi x')dx', \tag{12}$$

can be calculated by means of elementary functions and can be also expressed in the form of a series

$$\Psi(x,\varphi)=\frac{h_0^{(+)}}{\varphi}[1-\exp(-\varphi x)]+\Omega(x,\varphi)\exp(-\varphi x)-\Omega(0,\varphi), \tag{13}$$

where

$$\Omega(x,\varphi)=\sum_{k=1}\{\sigma_{2k}\sin(2kx)+\xi_{2k}\cos(2kx)\} \tag{14}$$

with coefficients

$$\sigma_{2k}=\frac{2kc_{2k}^{(+)}-\varphi s_{2k}^{(+)}}{\varphi^2+(2k)^2},$$

$$\xi_{2k}=-\frac{2ks_{2k}^{(+)}+\varphi c_{2k}^{(+)}}{\varphi^2+(2k)^2}. \tag{15}$$

The solution of Eq. (9) then reads

$$P(x;\varphi)=H^{(-)}(x)\left\{P_0 e^{\varphi x}-j\left[\frac{h_0^{(+)}}{\varphi}(e^{\varphi x}-1)+\Omega(x;\varphi)+\Omega(0;\varphi)e^{\varphi x}\right]\right\}. \tag{16}$$

With this formula at hand, we can calculate the two unknown quantities P_0 and j using two conditions (periodicity and normalization):

$$P(0;\varphi)=P(\pi;\varphi), \tag{17a}$$

$$\int_0^{\pi} P(x)dx=1. \tag{17b}$$

Therefore, within arbitrarily good accuracy the probability and the current are given by

$$P(x;\varphi)=jH^{(-)}(x)\left\{\frac{h_0^{(+)}}{\varphi}-\Omega(x;\varphi)\right\}, \tag{18}$$

$$j(\varphi)=\frac{\varphi}{\pi h_0^{(-)}h_0^{(+)}-\varphi\Phi(\varphi)}, \tag{19}$$

respectively. The integral

$$\Phi(\varphi)=\int_0^{\pi} H^{(-)}(x)\Omega(x,\varphi)dx \tag{20}$$

can be also represented as the series

$$\Phi(\varphi)=\frac{\pi}{2}\sum\ [s_{2k}^{(-)}\sigma_{2k}(\varphi)+c_{2k}^{(-)}\xi_{2k}(\varphi)]. \tag{21}$$

The structure of coefficients allows for splitting Φ into even and odd parts $\Phi=\pi(\Phi_{even}+\Phi_{odd})$:

$$\Phi_{even}=\sum_{k=1}\ \frac{k}{\varphi^2+(2k)^2}\{s_{2k}^{(-)}c_{2k}^{(+)}-c_{2k}^{(-)}s_{2k}^{(+)}\}, \tag{21a}$$

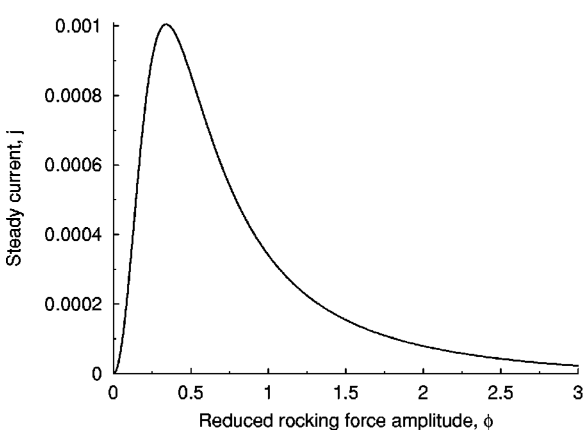

FIG. 2. Steady current $j=J/D_0$ [Eq. (19)] vs reduced rocking force amplitude $\phi=f/D_0$ for the asymmetric potential $V(x)$, Eq. (7), and order of expansion $n=6$. Parameters used: $D_0=0.1$, $\alpha=-0.3$, $\gamma=-0.2$, $V_0=1.5$.

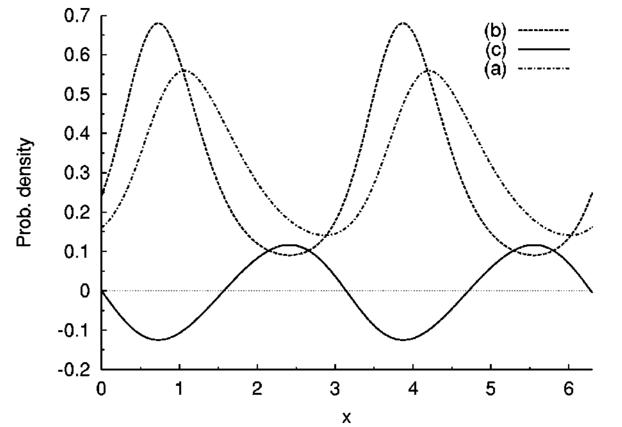

FIG. 3. Comparison of probability density with zero current (dash-dotted line) with probability density shifted in the presence of the net current (dashed line). (a) Boltzmann factor given by a starting potential, Eq. (7); (b) probability density shifted to the right in the presence of net ratchet current caused by a rocking force of amplitude $\phi=0.22$, Eq. (18); (c) the reduced potential function $V(x)/V_0$.

$$\Phi_{odd}=-\frac{1}{2}\sum\ \frac{\varphi}{\varphi^2+(2k)^2}\{s_{2k}^{(-)}s_{2k}^{(+)}+c_{2k}^{(-)}c_{2k}^{(+)}\}, \tag{21b}$$

and Eq. (19) can be rewritten in a more symmetric form

$$j=\frac{1}{\pi}\frac{\varphi^2\Phi_{even}(\varphi)}{\{h_0^{(-)}h_0^{(+)}-\varphi\Phi_{odd}(\varphi)\}^2-\{\varphi\Phi_{even}(\varphi)\}^2}. \tag{22}$$

The dependence of the net flux j Eq. (19) or (22), as a function of the reduced amplitude of the rocking force ϕ is presented in Fig. 2. Several calculations show that the more symmetrical and smooth the potential is, the lesser is the effect of the net current. In principle, the steady current given by Eq. (19) can become infinite when a particular combination of the expansion coefficients in the denominator becomes zero. We investigated this possibility for several combinations of parameters [cf. Eq. (9)] but have not detected this irregularity. The influence of the shape function and different range of parameters on the probability density function, thus ultimately on values of the current, which could lead to a "resonance" singularity is postponed for further investigation.

Figure 3 shows how the probability density function shifts in a (slightly) asymmetric potential when a net current flows

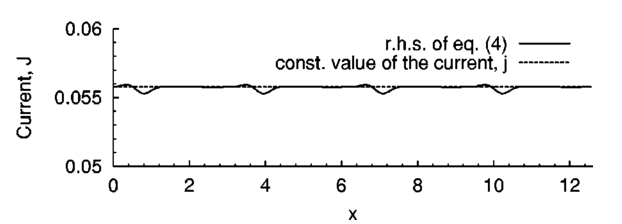

FIG. 4. Check of accuracy of the solution of Eq. (4). A constant value of a current j (straight line) as compared with the right side of Eq. (4) (parameters used have the same values as those for Fig. 2).

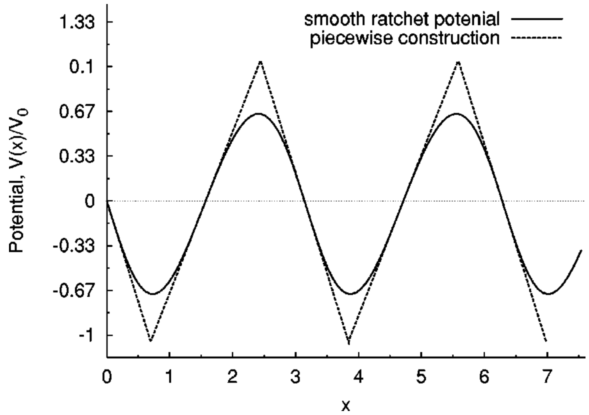

FIG. 5. A rough approximation of the smooth ratchet potential ratchet, $V(x)/V_0$, as a function of position x, with a piecewise construction considered by Magnasco. For comparison, both potentials have the same slopes rather than amplitudes. $\alpha=-0.3$, $\gamma=-0.2$, $V_0=1.5$.

[the right-shifted curve described by Eq. (18)]. The present approximation to the calculation of smooth ratchet effects seems to be quite reasonable. In Fig. 4 the reduced form of Eq. (4) (meaning all quantities are normalized by D_0) is plotted. The only (small) discrepancies between the left side (constant) and right side (which still is a function of x) are noticeable in places of potential twists. The stability of the calculated net current depends on the number of terms taken in the series expansion, and it also depends on how symmetric, or asymmetric, is the potential. For a more symmetric potential, the current approaches a constant value faster than for an asymmetric one, however in both cases the inclusion of only few terms gives a good approximation.

To have an idea of how details of ratchet potentials affect the overall effects of unidirectional motions, we tried to compare our results with those given by Magnasco [9]. Already at the beginning, when we start from ratchet potentials, no straightforward comparison is possible: we have to take either the same amplitudes or the same tilts. It is much more reasonable to assume the same slopes and the same periods

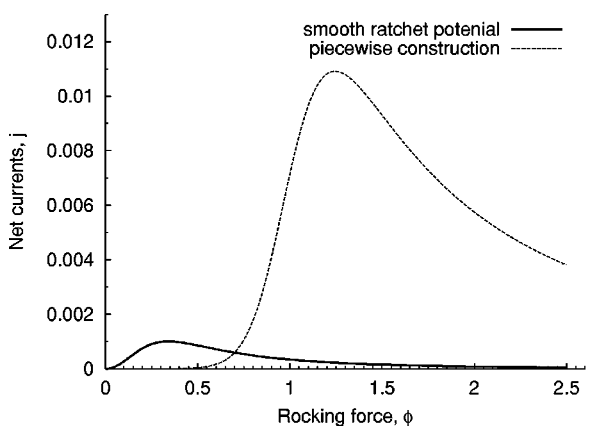

FIG. 6. Comparison of net currents $j=J/D_0$ for compatible piecewise and smooth potentials (cf. Fig. 5). The smoothness of the potential reduces significantly the net current effect (up to several orders of magnitude).

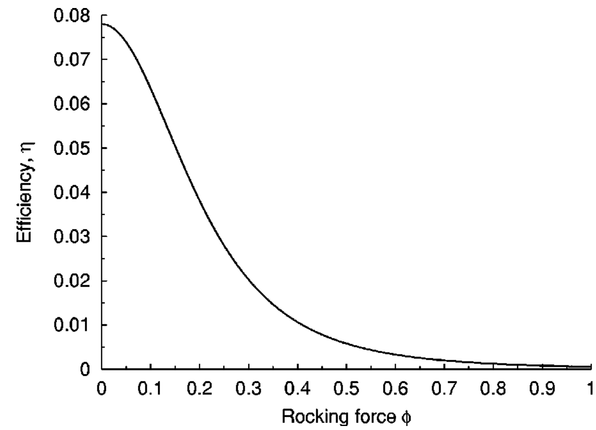

FIG. 7. Efficiency η of the smooth ratchet as a function of reduced rocking force $\phi=f/D_0$.

(with a greater amplitude) as demonstrated in Fig. 5, than amplitudes. We obtain the net current almost two orders smaller for a smooth ratchet than in the case of a piecewise potential function (Fig. 6). It is noticeable again how the smoothness of the potential radically diminishes the net current. With an expression for the current at hand, we can also calculate the efficiency of the smooth ratchet. A handy expression for efficiency is given in Ref. [16],

$$\eta=\frac{1}{\varphi}\frac{1-|j(-\varphi)/j(\varphi)|}{1+|j(-\varphi)/j(\varphi)|}. \tag{23}$$

The resulting efficiency is shown in Fig. 7. One can see that the efficiency of this kind of rocked ratchet is quite small, in agreement with the results obtained by other workers.

IV. CONCLUSIONS

In this paper we presented an alternative approach to Magnasco's "exact" solution of the ratchet potential model [9]. We chose a particular type of a potential function that would allow for a series expansion due to the shape parameter. This function is of logarithmic type. Its shape can be justified on grounds similar to those used by Braun [12] in describing two-dimensional diffusion on a surface: logarithmic potential functions are characteristic for "entropic" ratchets.

It seems that the entropic, rocked ratchets suffer from notoriously low efficiency [17]. This is confirmed also in the present investigation—smooth potentials cannot provide big net currents. In this respect, it appears that ratchets with a position-dependent diffusivity (or multiplicative noise) may yield much more efficient devices.

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APPENDIX A

Let us note that smooth potentials met in ratchet problems are frequently related to logarithmic functions. For example, an asymmetric potential of the "tooth" used by Magnasco and Stolovitzky [18],

$$V_{MS}(x)=V_0\frac{\sin(x)\cos(x)}{1+\gamma\cos^2(x)}, \tag{A1}$$

is obtained as a derivative of a logarithmic function:

$$V_{MS}(x)=\frac{V_0}{2\gamma}\frac{d}{dx}\ln[1+\gamma\cos^2(x)]. \tag{A2}$$

Another popular ratchet potential

$$V(x)=V_0\left\{\sin(x)+\frac{1}{4}\sin(2x)\right\} \tag{A3}$$

can be viewed as a sum of first two terms in a Fourier series that appears to be the Clausen function (an integral of the logarithm) [15],

$$-\int_0^{2x}\ln[2\sin(\sigma)]d\sigma=\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k^2}, \tag{A4}$$

$$V_C(x)=\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k^2}. \tag{A5}$$

The force for this potential reads

$$F(x)=\ln|\sin(2x)|+2\ln 2. \tag{A6}$$

Although it can be written as a simple elementary function, there is no real analytical benefit, for the Boltzmann factor is still given by

$$\exp\left(-\int\ln[\sin(x)]\right).$$

Another simpler function, very similar in shape to the Clausen function, is

$$V(x)=V_0\tan(x)\ln[\sin^2(x)], \tag{A7}$$

or, in a more symmetric form,

$$V(x)=V_0\frac{d}{dx}\{\ln[\sin^2(x)]\ln[\cos^2(x)]\}. \tag{A8}$$

Although the potentials (A7) and (A8) look rather simple, their application is limited since they produce singularities in forces and probabilities.

APPENDIX B

The coefficients appearing in Eq. (11) can be easily calculated by using a binomial formula. However, as there are positive and negative contributions, the general formulas are lengthy and contain many sums of sums. Therefore we provide here explicit formulas for coefficients up to order $O(\gamma^7)$:

$$s_2^{(-)}=\alpha\gamma(\mu)_1\left[1-\frac{1}{2}\gamma+\frac{5}{16}\gamma^2-\frac{7}{32}\gamma^3+\frac{21}{128}\gamma^4-\frac{33}{256}\gamma^5\right]+\frac{\alpha^3\gamma^3(\mu)_3}{8}\left[1-\frac{3}{2}\gamma-\frac{15}{8}\gamma^2\right]+\frac{\alpha^5\gamma^5(\mu)_5}{192}\left[1-\frac{5}{2}\gamma\right],$$

$$s_4^{(-)}=-\frac{\alpha\gamma^2}{4}\left\{(\mu)_1\left[1-\gamma+\frac{7}{8}\gamma^2-\frac{3}{4}\gamma^3+\frac{165}{256}\gamma^4\right]+\frac{\alpha^2\gamma^2(\mu)_3}{4}\left[1-2\gamma+\frac{45}{16}\gamma^2\right]+\frac{5\alpha^4\gamma^4(\mu)_5}{384}\right\},$$

$$s_6^{(-)}=\frac{\alpha\gamma^3}{16}\left\{(\mu)_1\left[1-\frac{3}{2}\gamma+\frac{27}{16}\gamma^2-\frac{55}{32}\gamma^3\right]-\frac{2\alpha^2(\mu)_3}{3}\left[1-\frac{3}{2}\gamma+\frac{9}{8}\gamma^2\right]-\frac{\alpha^2\gamma^2(\mu)_5}{48}\left[1-\frac{5}{2}\gamma\right]\right\},$$

$$s_8^{(-)}=-\frac{\alpha\gamma^4}{64}\left\{(\mu)_1\left[1-2\gamma+\frac{11}{4}\gamma^2\right]-2\alpha^2(\mu)_3\left[1-2\gamma+\frac{5}{2}\gamma^2\right]-\frac{\alpha^4\gamma^2(\mu)_5}{6}\right\},$$

$$s_{10}^{(-)}=\frac{\alpha\gamma^5}{256}\left\{[(\mu)_1-4\alpha^2(\mu)_3]\left[1-\frac{5}{2}\gamma\right]-\frac{15\alpha^4(\mu)_5}{2}\left[1-\frac{2}{45}\gamma\right]\right\},\qquad s_{12}^{(-)}=-\frac{\alpha\gamma^6}{1024}\left\{(\mu)_1-\frac{20\alpha^2(\mu)_3}{3}+\frac{2(\mu)_5}{3}\right\},$$

$$c_2^{(-)}=-\frac{\alpha^2\gamma^3}{8}\left\{(\mu)_2\left[1-\frac{3}{2}\gamma+\frac{7}{4}\gamma^2-\frac{5}{8}\gamma^3\right]+\frac{\alpha^2\gamma^2(\mu)_4}{12}\left[1-\frac{15}{2}\gamma\right]-\frac{\alpha^2\gamma^2(\mu)_6}{360}\right\},$$

$$c_4^{(-)}=\frac{\alpha^2\gamma^2}{4}\left\{(\mu)_2\left[1-\gamma+\frac{3}{4}\gamma^2-\frac{1}{2}\gamma^3+\frac{75}{256}\gamma^4\right]+\frac{\alpha^2\gamma^2(\mu)_4}{12}\left[1-2\gamma+\frac{85}{384}\gamma^2\right]+\frac{17\alpha^4\gamma^4(\mu)_6}{5760}\right\},$$

$$c_6^{(-)}=\frac{\alpha^2\gamma^3}{8}\left\{(\mu)_2\left[1-\frac{3}{2}\gamma+\frac{13}{4}\gamma^2-\frac{25}{16}\gamma^3\right]+\frac{\alpha^2\gamma^2(\mu)_4}{8}\left[1-\frac{5}{2}\gamma\right]-\frac{\alpha^4\gamma^3(\mu)_6}{240}\right\},$$

$$c_8^{(-)} = -\frac{3\alpha^2\gamma^4}{64}\left\{(\mu)_2\left[1-2\gamma+\frac{65}{24}\gamma^2\right]-\frac{\alpha^2(\mu)_4}{9}\left[1-2\gamma+\frac{5}{4}\gamma^2\right]-\frac{\alpha^4\gamma^2(\mu)_6}{540}\right\},$$

$$c_{10}^{(-)} = \frac{\alpha^2\gamma^5}{64}\left\{(\mu)_2\left[1-\frac{5}{2}\gamma\right]-\frac{\alpha^2(\mu)_4}{3}\left[1-\frac{5}{2}\gamma\right]-\frac{\alpha^2\gamma(\mu)_6}{90}\right\},$$

$$c_{12}^{(-)} = -\frac{\alpha^2\gamma^6}{1024}\left\{(\mu)_2+\frac{2\alpha^2}{3}(\mu)_4-\frac{2\alpha^4}{225}(\mu)_6\right\}.$$

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